MATH 223 Assignment #5

1. Let $A \in \mathbf{R}^{m\times n}$ have rank 1. Show that there exist non-zero vectors $x \in \mathbf{R}^m$ and $y \in \mathbf{R}^n$ so that $A = xy^T$. (Hint: Try a simple case and also compute xy^T for some simple choices for x and y.) (Comment: You could explore how to generalize such a result to higher rank.)

2. Determine bases for the following subspaces of $\mathbf{R}^3$.

   a) the line $x = 5t, y = -2t, z = t$.

   b) all vectors of the form $(a, b, c)^T$ such that $a - 3b = 2c$.

3. Let
$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & -3 & 1 \\ 0 & 2 & 0 & 6 & -6 & 0 \\ 0 & 3 & 7 & 2 & -9 & 7 \\ 0 & 2 & 2 & 4 & -4 & 3 \end{bmatrix}$$
   Determine a basis for the column space of A (chosen from columns of A) and determine a basis for the row space of A. Also give a basis for the *nullspace* of A, namely $\{\mathbf{x} \in \mathbf{R}^6 : A\mathbf{x} = \mathbf{0}\}$.

4. Show that the set of all vectors $(b_1, b_2, b_3, b_4)^T$ such that the system below is consistent (i.e. can be solved)
$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 3 & 3 \\ 1 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$
   is a subspace of $\mathbf{R}^4$. Then find a basis of the subspace.

5. Let A be an $n \times n$ matrix with various eigenvalues including λ and μ with $\lambda \neq \mu$. Let L, M be the eigenspaces associated with eigenvalues λ, μ respectively. (That is, L is the set of all eigenvectors with eigenvalue λ; M is the set of all eigenvectors with eigenvalue μ.) Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be a basis for L and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$ be a basis for M. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$ is a linearly independent set of $p + q$ vectors. (Hint: try $p = 1$ and $q = 1$ to start). (Comment: You could explore the case if there were three different eigenvalues and three bases for the eigenspaces).

6. Let $\mathbf{R}^{n\times n}$ denote the vector space of all $n \times n$ matrices (over $\mathbf{R}$). Consider following transformation $f : \mathbf{R}^{n\times n} \to \mathbf{R}^{n\times n}$
$$f(A) = A^T.$$
   Show that this is a linear transformation.

   We say that a matrix A is *symmetric* if $A^T = A$ and we say that a matrix A is *skew-symmetric* if $A^T = -A$.

   a) Warmup question: Give a basis for $\mathbf{R}^{n\times n}$. How many elements are in your basis?

   b) What is the dimension of the eigenspace of eigenvalue 1 for f? Explain.

   c) What is the dimension of the eigenspace of eigenvalue -1 for f? Explain.

   d) Now use the previous question (and other facts) to show that any $A \in \mathbf{R}^{n\times n}$ is a linear combination of a symmetric matrix and a skew-symmetric matrix (you could show this directly of course but I'm asking you to use linear independence/dimension arguments).